

## Take-Home Test 4, Math 240, Fall 2005

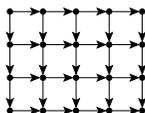
*Due: 2:45pm, November 15. Value: 50 pts.*

**Instructions:** The rules are the same as for Take-Home Test 3: Complete five of the following six problems. The only resources you may use are your textbook, your class notes, your own brain, and your instructor. You may also use a pencil and paper if you must.

You will have the opportunity to complete a “rewrite,” due the class day following when I distribute your initial grades. Because of the rewrite policy, you should be careful not to discuss the test with others or access related resources even *after* you have submitted your initial version.

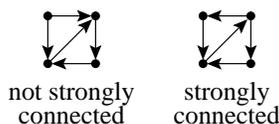
**Problem A.** What is the probability that two randomly chosen graphs on four vertices are isomorphic? To choose a random graph, we flip a fair coin for each of the six possible edges to see whether the edge is in the graph; it is possible that we could choose the same random graph twice. (Hint: There’s not a nice way for computing this to my knowledge. I listed all non-isomorphic four-vertex graphs as part of my answer.)

**Problem B.** Given two integers  $m$  and  $n$ , we construct a directed graph with  $mn$  vertices arranged in an  $m \times n$  grid, where each vertex is connected to the vertex below it and to its right. The following grid is the result if we have  $m = 4$  and  $n = 5$ .



Find an expression in terms of  $m$  and  $n$  for the number of paths in such a graph from the upper left vertex to the lower right vertex. Prove your answer using induction on either  $m$  or  $n$ .

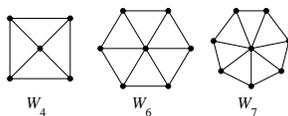
**Problem C.** A directed graph is *strongly connected* if there is a path from every vertex to all other vertices. Below, the graph at left is not strongly connected because there is no path from the lower right vertex to the upper left vertex.



Prove that if a graph with  $n$  vertices has less than  $n$  edges, then it is not strongly connected. (By contrast, an undirected graph with  $n - 1$  edges can still be connected.)

**Problem D.** Prove that for every graph that is not connected, its complement  $\overline{G}$  is connected.

**Problem E.** For every positive integer  $n$ , we can define  $W_n$  to be a *wheel* graph on  $n$  vertices: This graph consists of  $n$  vertices arranged in a cycle, with an additional vertex in the center connected to each of the other vertices.



Determine how  $W_n$ 's chromatic index is related to  $n$ , and prove your answer.

**Problem F.** Prove that for every graph, its chromatic index is  $n$  if and only if it is a complete graph ( $K_n$ ), where  $n$  represents the number of vertices in the graph.